



Geometry of Root-Related Parameters of PH Curves

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(Received November 2000; revised and accepted January 2002)

Abstract—We study the (plane polynomial) Pythagorean hodograph curves from the viewpoint of their roots. The loci of root-related parameters of PH curves show us very interesting geometric properties. They include regular $2n + 1$ -gon and isosceles triangles with the ratio of sides $n : 1 : n$.
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Keywords—Pythagorean hodograph curve, Hermite interpolation.

1. INTRODUCTION

Rational parametrization of curves and surfaces is a very interesting problem in mathematics. For curves, it is shown that a curve of genus zero is a rational by Noether. As for surfaces, there is Noether's theorem that an algebraic surface S is rational if and only if it contains an irreducible rational curve C with $\dim|C| \geq 1$. Although there exist some techniques to find a rational parametrization, such as Abhyankar's method for a quartic curve with three double points, moving line/surface method, and adjoint method, finding a rational parametrization in practice is not an easy task.

The main difficulties in rational parametrization of geometrically driven objects which are widely used in CAGD such as offset, pipe/canal surface, and swept volume are caused by the irrationality of unit normal vector fields. Once we restrict ourselves to rational curves and surfaces whose unit normal vectors are rational, many troubles disappear. This is the motivation of Pythagorean hodograph curves and Pythagorean normal surfaces. By its definition, a polynomial/rational curve is PH if its speed function is a polynomial/rational. After Farouki introduced Pythagorean hodograph curves in [1], there has been vast research on this class of curves by him and others [2–7].

As Hilgarter *et al.* [8] pointed out, the offset with varying distance function $r(t)$ given by the spine curve $m(t)$ admits a rational parametrization over \mathbb{R} in accordance with $m(t)$ if and only

if $(m(t), r(t))$ is an MPH curve in $\mathbb{R}^{2,1}$. Of course, if the radius function $r(t)$ is constant, then the offset is reduced to the classical offset and MPH condition is reduced to PH condition. For the canal surface which is the envelope of a one-parameter family of moving spheres, we know that the canal surface given by the spine curve m and the radius function r admits a rational parametrization over the reals in accordance with the spine curve m if and only if (m, r) is a space-like curve [8,9]. However, the computational cost of the rational parametrization algorithm [8,9] is too high. If we restrict the spine curve of canal surface to the MPH curve which was introduced by Moon [10] and was used to compute the medial axis transform by Choi *et al.* [11], then computational cost is reduced.

In this paper, we recharacterize the Pythagorean hodograph curves. All plane PH curves are determined by the zeros of their complexified hodograph curves. This means that all possible PH curves are classified algebraically. However, the geometric properties are not clarified yet. Farouki showed some geometric properties of PH curves of low degrees such as ratios of and angles between legs of control polygon.

As an application of the above result, we show that the loci of root-related parameters of regular PH curves have special geometric properties including interesting basic forms such as regular n -gons and isosceles triangles. We believe that our geometric characterization using roots of PH curves is more lucid and gives a way of geometric classification of PH curves. Moreover, we solve a C^1 Hermite interpolation problem with regular PH quintic in our root-representation of PH curves. As a result, we get four interpolants and it coincides with Farouki's result [12].

2. NECESSARY AND SUFFICIENT CONDITION TO BE PYTHAGOREAN

DEFINITION 2.1. Two complex numbers z_1, z_2 are *semiequal* if $z_1 = z_2$ or $z_1 = \bar{z}_2$, (denote by $z_1 \approx z_2$) and z_1, z_2 are *distinct up to conjugate* if z_1, z_2 are not semiequal.

DEFINITION 2.2. A plane polynomial curve $\alpha(t) = (a(t), b(t))$ is *Pythagorean* if $a(t)^2 + b(t)^2 = c(t)^2$ for some polynomial $c(t)$. A plane polynomial curve $\alpha(t)$ has *Pythagorean hodograph* if its hodograph $\alpha'(t)$ is Pythagorean.

We need the following proposition [13] which can be proved by factorization over \mathbb{C} .

PROPOSITION 2.3. For a complexified polynomial curve $z(t)$ of degree n , $z(t)$ is Pythagorean if and only if n roots of $z(t) = 0$ consist of only real roots and pairs of semiequal roots.

3. GEOMETRY OF ROOT-RELATED PARAMETERS OF PH CURVES

We apply the factorization method for complexified curves, not to their hodographs, but to original curves. Let $\alpha(t) = a(t) + ib(t)$ be a complexified plane curve. By the fundamental theorem of algebra, α can be rewritten as $\alpha = K \prod_{i=1}^n (t - c_i)$ where K and c_i are complex constants. Now, we can characterize regular PH curves via roots as follows.

PROPOSITION 3.1. If α is a regular PH curve, that is, the component functions of its hodograph $\alpha'(t) = (a'(t), b'(t))$ are relatively prime:

$$\alpha'(t) = K \prod_{j=1}^N (t - e_j)^{2s_j},$$

then the following is true.

- (i) $\alpha(t)$ cannot have any real zero of multiplicity greater than 1.
- (ii) If c is a nonreal complex zero of $\alpha(t)$, then multiplicity of c is odd.
- (iii) Of all zeros of $\alpha(t)$, the number of distinct zeros are odd.

(iv) Let c_i be root of $\alpha(t) = 0$ of multiplicity s_i for $i = 1, 2, \dots, 2m - 1$.

$$d(t) = \sum_{i=1}^{2m-1} s_i (t - c_1) \cdots \widehat{(t - c_i)} \cdots (t - c_{2m-1}) \quad (3.1)$$

is constant or roots of $d(t) = 0$ are nonreal complex numbers of even multiplicities and are not semiequal to any of themselves nor c_i for $s_i \geq 2$.

PROOF. Taking multiplicities of zeros into account, we can rewrite $\alpha(t)$ as $\alpha(t) = K \prod_{i=1}^l (t - c_i)^{s_i}$. Then its hodograph is

$$\alpha'(t) = K \left(\prod_{i=1}^l (t - c_i)^{s_i-1} \right) \left(\sum_{i=1}^l s_i (t - c_1) \cdots \widehat{(t - c_i)} \cdots (t - c_l) \right), \quad (3.2)$$

where $\widehat{(t - c_i)}$ means that it is omitted.

- (i) If there exists a real zero of multiplicity 2, then it is also a zero of $\alpha'(t)$ and it leads to a contradiction to regularity condition.
- (ii) If a nonreal complex number c is a zero of multiplicity even, then c appears odd times in the second factor $(\prod_{i=1}^l (t - c_i)^{s_i-1})$ in the right-hand side of (3.2) and c_i s are distinct from \bar{c}_j because $\alpha'(t)$ does not have a real polynomial as its factor. Thus, for $\alpha(t)$ to be regular, c_j is a zero of the third factor $\sum_{i=1}^l s_i (t - c_1) \cdots \widehat{(t - c_i)} \cdots (t - c_l)$. Hence, we get

$$\sum_{i=1}^l s_i (t - c_1) \cdots \widehat{(t - c_i)} \cdots (t - c_l) \Big|_{t=c_j} = s_j \prod_{i \neq j, i=1}^l (c_j - c_i) = 0.$$

This implies that s_j is zero and it leads to a contradiction to the multiplicity of $c_j \geq 2$.

- (iii) If there are an even number of distinct zeros, then by (i) and (ii), $K(\prod_{i=1}^l (t - c_i)^{s_i-1})$ is a curve of even degree whose zeros are only real and semiequal complex roots. By Proposition 2.3, it is a Pythagorean curve. For $\alpha(t)$ to be PH, $\sum_{i=1}^l (t - c_1) \cdots \widehat{(t - c_i)} \cdots (t - c_l)$ must be Pythagorean. However, it is impossible because the degree is odd.
- (iv) It is clear from Proposition 2.3. ■

Proposition 3.1 means that for plane polynomial curve $\alpha(t)$ to be a regular PH curve, the zeros of $\alpha(t)$ have a special property. In particular, the last condition of the proposition shows that the special relation between distinct roots of $d(t) = 0$ determines whether $\alpha(t)$ is a regular PH curve. In the following example, we will see that the distinct roots have a special geometric property.

EXAMPLE 3.2. Let $\alpha(t) = a(t) + ib(t)$ be a regular PH cubic curve. By the regularity condition, $\alpha'(t)$ is either $K(t - c)(t - \bar{c})$ or $K(t - c)^2$ for some nonreal complex number c . If $\alpha'(t) = K(t - c)(t - \bar{c})$, then $\alpha(t)$ is a line. In this case, the component functions of the hodograph are not relatively prime and we omit this case. If the hodograph is given by $K(t - c)^2$, then $\alpha(t) = (1/3)K(t - c)^3 + z_0$ for some complex constant z_0 and in the complex plane, the locus of roots of $\alpha(t) = 0$ is either a three-fold point (if $z_0 = 0$) or three vertices of a regular triangle such that any pair of vertices are not conjugate (if $z_0 \neq 0$).

Although it is hard to analyze the structure of roots of general PH curves of high degree, two special cases are tractable, as shown in the following two corollaries.

COROLLARY 3.3. ISOSCELES TRIANGLE. Suppose $\alpha(t)$ is a PH curve of degree $2n + 1$ whose hodograph is

$$\alpha'(t) = (2n + 1)K(t - c_1)^{2n-2}(t - c_2)^2. \quad (3.3)$$

Then $\alpha(t)$ is given by

$$\alpha(t) = K(t - c_1)^{2n-1}(t - \delta_1)(t - \delta_2) + k,$$

for some complex numbers δ_1, δ_2, k such that the triangle $\triangle c_1 \delta_1 \delta_2$ is an isosceles triangle with the ratio of sides $n : 1 : n$.

PROOF. If we divide $\alpha(t)$ by $K(t - c_1)^{2n-1}$, then we get the following relations:

$$\alpha(t) = K(t - c_1)^{2n-1} (t - \delta_1)(t - \delta_2) + R_{2n-2}(t),$$

where δ_1, δ_2 are complex numbers and $R_{2n-2}(t)$ is the remainder polynomial of degree $2n-2$. Since the polynomial $R'_{2n-2}(t)$ of degree $2n-3$ is divided by the polynomial $(t - c_1)^{2n-2}$, $R'_{2n-2}(t) = 0$, and hence, $R_{2n-2}(t) \equiv k$. Translating by $-\alpha(0)$, we may assume $k = 0$. Thus,

$$\alpha'(t) = K(t - c_1)^{2n-2} [(2n-1)(t - \delta_1)(t - \delta_2) + (t - c_1)(t - \delta_1) + (t - c_1)(t - \delta_2)].$$

Comparing the above equation and (3.3), we get

$$(2n-1)(t - \delta_1)(t - \delta_2) + (t - c_1)(t - \delta_1) + (t - c_1)(t - \delta_2) = (2n+1)(t - c_2)^2,$$

equivalently,

$$(2n-1)(\delta_1 + \delta_2) + 2c_1 + \delta_1 + \delta_2 = 2(2n+1)c_2, \quad (3.4)$$

$$(2n-1)\delta_1\delta_2 + c_1(\delta_1 + \delta_2) = (2n+1)c_2^2. \quad (3.5)$$

Thus,

$$\frac{c_1 - \delta_1}{\delta_2 - \delta_1} = n^2 \frac{\delta_1 - \delta_2}{c_1 - \delta_2}.$$

This identity implies that the angle $\angle c_1 \delta_1 \delta_2$ is equal to $\angle c_1 \delta_2 \delta_1$. Thus, the triangle $\triangle c_1 \delta_1 \delta_2$ is an isosceles triangle. The ratio of sides of the triangle $\triangle c_1 \delta_1 \delta_2$ is $\overline{c_1 \delta_1} : \overline{\delta_1 \delta_2} : \overline{\delta_2 c_1} = n : 1 : n$. Thus, the roots of $\alpha(t)$ form an isosceles triangle. See Figure 1a. \blacksquare

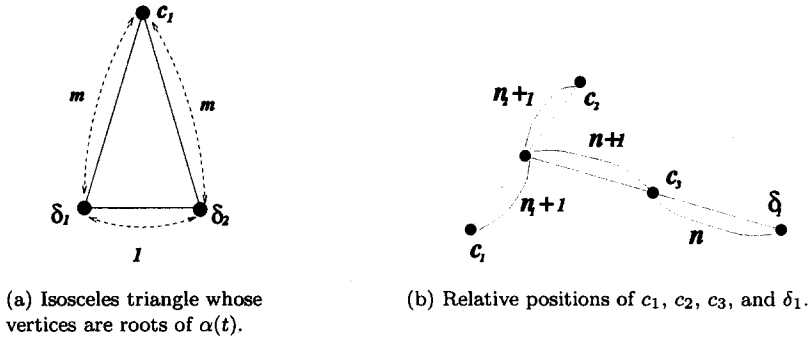


Figure 1.

COROLLARY 3.4. Let $\alpha(t)$ be a PH curve of degree $2n+1$ whose hodograph is given by

$$\alpha'(t) = (2n+1)K(t - c_1)^{2n_1}(t - c_2)^{2n_2}(t - c_3)^2, \quad (3.6)$$

where n_1, n_2 are nonnegative integers with $n_1 + n_2 + 1 = n$ and c_1, c_2, c_3 are complex numbers. Then $\alpha(t)$ is given by

$$\alpha(t) = K(t - c_1)^{2n_1+1}(t - c_2)^{2n_2+1}(t - \delta_1) + R_{2n-1}(t),$$

where δ_1 is a complex number and $R_{2n-1}(t)$ is a regular PH curve of degree $2n-1$ and $R'_{2n-1}(t) = M(t - c_1)^{2n_1}(t - c_2)^{2n_2}$. δ_1 and M are determined by the following equations:

$$c_3 = \frac{(n_2 + 1)c_1 + (n_1 + 1)c_2 + n\delta_1}{2n + 1},$$

$$\frac{M}{K} = \frac{((n_2 + 1)c_1 + (n_1 + 1)c_2 + n\delta_1)^2}{2n + 1} - ((2n_1 + 1)c_2 + (2n_2 + 1)c_1)\delta_1 - c_1c_2.$$

PROOF. Since degree of $\alpha(t)$ is $2n + 1$, we can write $\alpha(t)$ as follows:

$$\alpha(t) = K(t - c_1)^{2n_1+1}(t - c_2)^{2n_2+1}(t - \delta_1) + R_{2n-1}(t),$$

where $R_{2n-1}(t)$ is a polynomial of degree $2n - 1$. Since $\alpha'(t)$ is divisible by $(t - c_1)^{2n_1}(t - c_2)^{2n_2-2n_1-2}$ and $R'_{2n-1}(t)$ is of degree $2n - 2$, $R'_{2n-1}(t) = M(t - c_1)^{2n_1}(t - c_2)^{2n_2}$. So, $R_{2n-1}(t)$ is a PH curve of degree $2n - 1$. Thus,

$$\begin{aligned} \alpha'(t) &= K(t - c_1)^{2n_1}(t - c_2)^{2n_2} \\ &\times \left\{ (2n_1 + 1)(t - c_2)(t - \delta_1) + (2n_2 + 1)(t - c_1)(t - \delta_1) + (t - c_1)(t - c_2) + \frac{M}{K} \right\}. \end{aligned}$$

Comparing the above equation and equation (3.6), we get

$$\begin{aligned} (2n + 1)(t - c_3)^2 &= (2n_1 + 1)(t - c_2)(t - \delta_1) \\ &+ (2n_2 + 1)(t - c_1)(t - \delta_1) + (t - c_1)(t - c_2) + \frac{M}{K}, \end{aligned}$$

equivalently,

$$2(2n + 1)c_3 = (2n_1 + 1)(c_2 + \delta_1) + (2n_2 + 1)(c_1 + \delta_1) + c_1 + c_2, \quad (3.7)$$

$$(2n + 1)c_3^2 = (2n_1 + 1)c_2\delta_1 + (2n_2 + 1)c_1\delta_1 + c_1c_2 + \frac{M}{K}. \quad (3.8)$$

Thus, we get

$$\begin{aligned} c_3 &= \frac{(n_2 + 1)c_1 + (n_1 + 1)c_2 + n\delta_1}{2n + 1}, \\ \frac{M}{K} &= \frac{((n_2 + 1)c_1 + (n_1 + 1)c_2 + n\delta_1)^2}{2n + 1} - ((2n_1 + 1)c_2 + (2n_2 + 1)c_1)\delta_1 - c_1c_2. \end{aligned}$$

As shown in the above two corollaries, in some cases, we get the geometric behaviours of roots themselves, but in other cases, we get *root-related parameters* c_i, δ_i . These parameters completely determine PH curves and they show interesting properties. For example, in the quintic case, an isosceles triangle can be obtained from the initial regular pentagon with vertices $c_j, j = 1, \dots, 5$, indexed counter-clockwisely which are roots of the simplest PH curve $K(t - c)^5 + z_0$ by pulling out a vertex (for convenience, we denote this vertex by c_1) with fixing c_3 and c_4 and no change of the length of each edge until the vertex $c_2(c_5)$ lies on the line segment $\overline{c_1c_3}(\overline{c_1c_4})$, respectively. In general, for regular PH curves of degree $2n + 1$, we can obtain isosceles triangle from the regular $(2n + 1)$ -gon in the same way.

3.1. PH Curves of Low Degree

In this section, we see the distribution of roots of regular PH curves of low degree, 5, 7. For a regular PH quintic, by the regularity condition, the possible cases for the hodograph $\alpha'(t)$ of $\alpha(t)$ are

- (i) $5K(t - c)^4$,
- (ii) $5K(t - c)^2(t - d)^2$.

In Case (i), it is clear that the roots of $\alpha(t) = K(t - c)^5 + z_0 = 0$ form a regular pentagon whose five vertices are lying on the perimeter of a circle of radius $\|z_0/K\|^{1/5}$ centered at c . In Case (ii), by Corollary 3.3, we know that the roots c, δ_1, δ_2 of $\alpha(t) = K(t - c)^3(t - \delta_1)(t - \delta_2) = 0$ form an isosceles triangle with the ratio of sides $2 : 1 : 2$.

For a regular PH curve of degree 7, the possible cases for the hodograph $\alpha'(t)$ are

- (i) $7K(t - c)^6$,
- (ii) $7K(t - c_1)^4(t - c_2)^2$,
- (iii) $7K(t - c_1)^2(t - c_2)^2(t - c_3)^2$.

In Cases (i) and (ii), it is easy to see that the roots of $\alpha(t)$ form a regular heptagon and an isosceles triangle, respectively. In Case (iii), by Corollary 3.4, the relative position of c_1, c_2, c_3, δ_1 is (see Figure 1b with $n_1 = 1, n_2 = 1$)

$$c_3 = \frac{2c_1 + 2c_2 + 3\delta_1}{7},$$

where $\alpha(t) = K(t - c_1)^3(t - c_2)^3(t - \delta_1) + \int M(t - c_1)^2(t - c_2)^2 dt$.

REMARK 3.5. We saw two and three cases for regular PH curves of degree 5 and 7, respectively. However, the last cases $\prod_i (t - c_i)^2$ are general cases and the preceding cases are special cases. For instance, if we impose $c_1 = c_2$ on isosceles case, then we get the regular n -gon case: from (3.4) and (3.5), we get

$$\delta_1 + \delta_2 = 2c_1, \quad \delta_1 \delta_2 = c_1^2.$$

Thus, $\delta_1 = \delta_2 = c_2 = c_1$ and this case is reduced to the regular n -gon case.

4. HERMITE INTERPOLATION

In this section, we try to apply our result on distribution of roots of regular PH curves to the Hermite interpolation problem. We know that, for C^1 Hermite data, a regular PH quintic is required. Thus, we are seeking a regular PH quintic curve $\alpha(t)$ such that

$$\alpha(0) = 0, \quad \alpha'(0) = v_0, \quad (4.1)$$

$$\alpha(1) = \alpha_1, \quad \alpha'(1) = v_1. \quad (4.2)$$

From (3.3), (4.1), and (4.2), we must solve the following system of equations with unknown variables $c_1, c_2, \delta_1, \delta_2, K, z_0$:

$$\alpha(0) = -Kc_1^2\delta_1\delta_2 + z_0 = 0, \quad \alpha(1) = K(1 - c_1)^3(1 - \delta_1)(1 - \delta_2) + z_0 = \alpha_1, \quad (4.3)$$

$$\alpha'(0) = 5Kc_1^2c_2^2 = v_0, \quad \alpha'(1) = 5K(1 - c_1)^2(1 - c_2)^2 = v_1, \quad (4.4)$$

$$c_2 = \frac{c_1 + 2\delta_1 + 2\delta_2}{5}, \quad (c_1 - \delta_1)(c_1 - \delta_2) = -4(\delta_1 - \delta_2)^2. \quad (4.5)$$

From (4.5), we get

$$\delta_1 + \delta_2 = \frac{5c_2 - c_1}{2}, \quad \delta_1\delta_2 = \frac{c_1^2 - 5c_1c_2 + 10c_2^2}{6}. \quad (4.6)$$

From (4.3), (4.4), and (4.6), we get

$$(1 - c_1)^3(1 - (\delta_1 + \delta_2)) + \delta_1\delta_2(c_1^3 + (1 - c_1)^3) = \frac{\alpha_1}{K} \Rightarrow$$

$$6 - 15(c_1 + c_2) + 20c_1c_2 + 10(c_1 + c_2)^2 - 30c_1c_2(c_1 + c_2) + 30c_1^2c_2^2 = \frac{30\alpha_1c_1^2c_2^2}{v_0}.$$

From (4.4), we get

$$\frac{1 - c_1}{c_1} \frac{1 - c_2}{c_2} = \left(\frac{v_1}{v_0} \right)^{1/2}.$$

If we let $\mu = c_1 + c_2$, $\nu = c_1c_2$, then we get the following system of one quadratic equation and one linear equation in μ and ν :

$$10\mu^2 - 30\mu\nu - 30 \left(1 - \frac{\alpha_1}{v_0} \right) \nu^2 - 15\mu + 20\nu + 6 = 0,$$

$$\left(1 - \left(\frac{v_1}{v_0} \right)^{1/2} \right) \nu - \mu + 1 = 0.$$

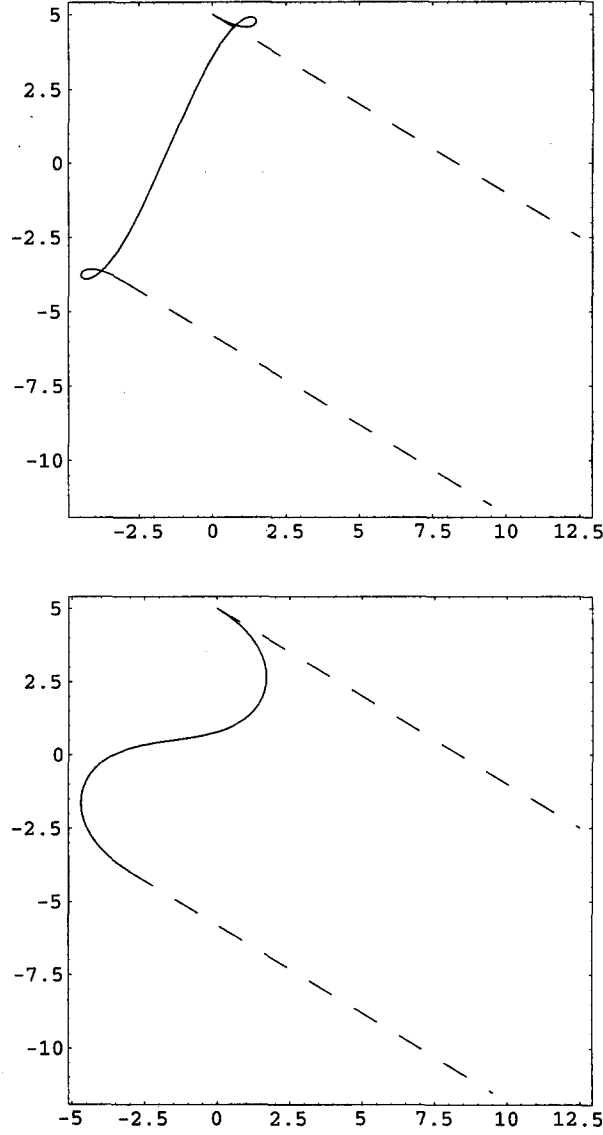


Figure 2. Four regular PH quintic interpolants with $\alpha_1 = -3 - 9i$, $v_0 = 25 - 15i$, $v_1 = 25 - 15i$. Solid curves are interpolants and dashed lines are input data.

Thus, we get two pairs of μ and ν , and hence, two interpolants $\alpha(x) = v_0/5\nu^2 \int (x^2 - \mu x + \nu)^2 dx$. Finally, considering the sign of $(v_1/v_0)^{1/2}$, we get four interpolants and it coincides with Farouki's result [12]. Figure 2 shows four quintic Hermite interpolants with initial data

$$\begin{aligned} \alpha(0) &= (0, 5), & \alpha'(0) &= (25, -15), \\ \alpha(1) &= (-3, -4), & \alpha'(1) &= (25, -15). \end{aligned}$$

Dotted lines are tangent vectors and solid curves are interpolating quintic curves.

4.1. Hermite Interpolation with Higher Degree Curves

We can solve the C^1 Hermite interpolation problem using PH curves of $2n + 1 \geq 5$ whose roots form an isosceles triangle. In other words, we can find a PH curve $\alpha(t)$ of degree $2n + 1$ given by $\alpha'(t) = (2n + 1)K(t - c_1)^{2n-2}(t - c_2)^2$.

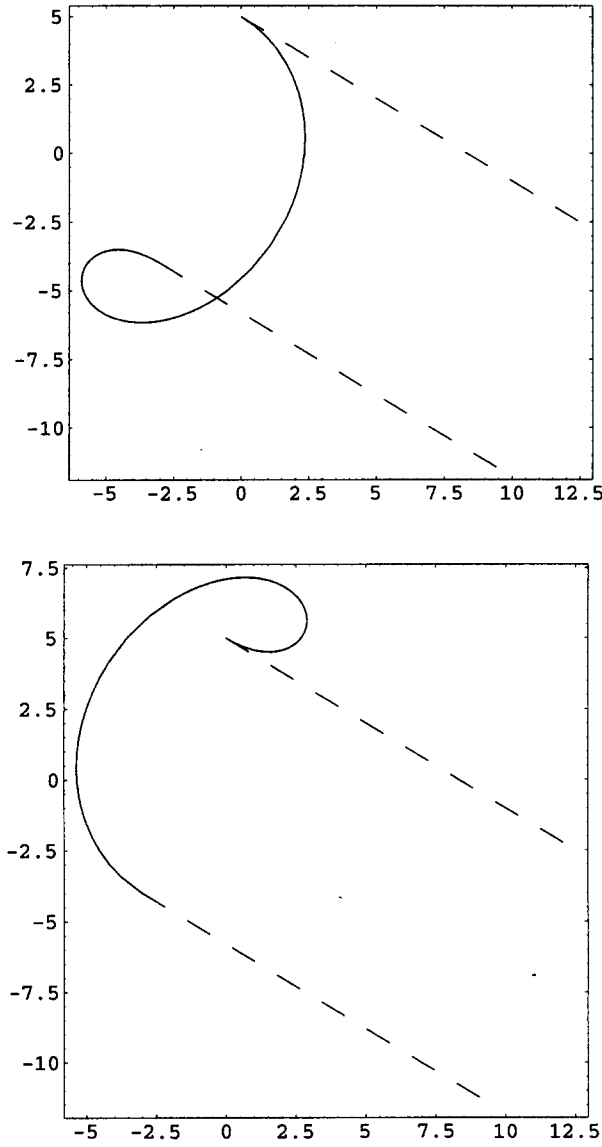


Figure 2. (cont.)

After some manipulations on constraints which are similar to the regular PH quintic case, we get the following two constraints:

$$\begin{aligned} (2n+1) \left(1 - \frac{\alpha_1}{v_0}\right) c_1^{2n-2} c_2^2 + \text{lower degree terms} &= 0, \\ \left(1 - \frac{v_1}{v_0}\right) c_1^{2n-2} c_2^2 + \text{lower degree terms} &= 0. \end{aligned}$$

By Bézout's theorem [14], the number of common solutions of the above two equations is $(2n) \times (2n) = 4n^2$ in a projective plane considering the multiplicities. However, since the number of solutions on line at infinity— $(1, 0, 0)$ and $(0, 1, 0)$ —are $2 \times 2 + 2 \times 2 = 8$, the number of solutions in the complex plane is $4n^2 - 8$. In the case of $n = 2$ (quintic), we can reduce this bound by half because of the symmetry of the equations. So the bound becomes $2n^2 - 4 = 4$, which is the exact number of solutions seen in the previous section.

5. CONCLUDING REMARK

We investigated the PH curves from the viewpoint of their roots. We found that the roots of (regular) PH curves should satisfy some conditions. Moreover, the loci of the roots of PH curves show us very interesting geometric objects such as regular n -gons, isosceles triangles, and four points with special distance ratios. As for regular PH curves of high degree, they are represented by a combination of basic shapes—regular n -gon, isosceles triangle, and special four points. We also solved C^1 Hermite interpolation in our scheme and get four interpolants.

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